ungraded

$\Sigma_{0,3}$

$\pi = \pi_1(\Sigma_{0,3}, *) = \langle \gamma_1, \gamma_2 \rangle$

$\widehat{\mathbb{K}\pi} \leftarrow$ Hopf alg

$|\widehat{\mathbb{K}\pi}| = \mathbb{K}\pi / [\mathbb{K}\pi, \mathbb{K}\pi] \cong \mathbb{K}(\pi/\text{conj})$

$xy - yx = 0$ ↖ Lie bialg

$x, y \in \pi$ $\quad xyx^{-1} = y$

graded

$H = H_1(\Sigma_{0,3}, \mathbb{K}) = \mathbb{K}z_1 \oplus \mathbb{K}z_2$

$z_1 = [\gamma_1]$, $z_2 = [\gamma_2]$

$g_j \neq 0$

$\Delta(g_j) = g_j \otimes g_j$

Hopf alg $\quad \Delta$

$\downarrow$

$\widehat{\mathbb{K}\pi} \longleftrightarrow \hat{T}(H)$

$$\boxed{2.5 \quad \exists g_j \in \mathrm{Grp}(\hat{T}(H)) \quad \theta(\gamma_j) = g_j^{-1} e^{z_j} \cdot g_j}$$

$|\widehat{\mathbb{K}\pi}| \longleftrightarrow |\hat{T}(H)| \cong \mathbb{K}\left\{\begin{matrix}\text{cyclic words}\\ \text{in } z_1, z_2\end{matrix}\right\}$

↖ Lie bialg

expansions

$\theta : \widehat{\mathbb{K}\pi} \longrightarrow \hat{T}(H)$

Conditions

1. θ is a Hopf alg hom.
2. $\mathrm{gr}\,\theta = \mathrm{id}$
3. $\theta : |\widehat{\mathbb{K}\pi}| \longrightarrow |\hat{T}(H)|$ respects

   ① $[\cdot,\cdot]_G$ & $[\cdot,\cdot]_{gr}$

   ② δ_T & δ_{gr}

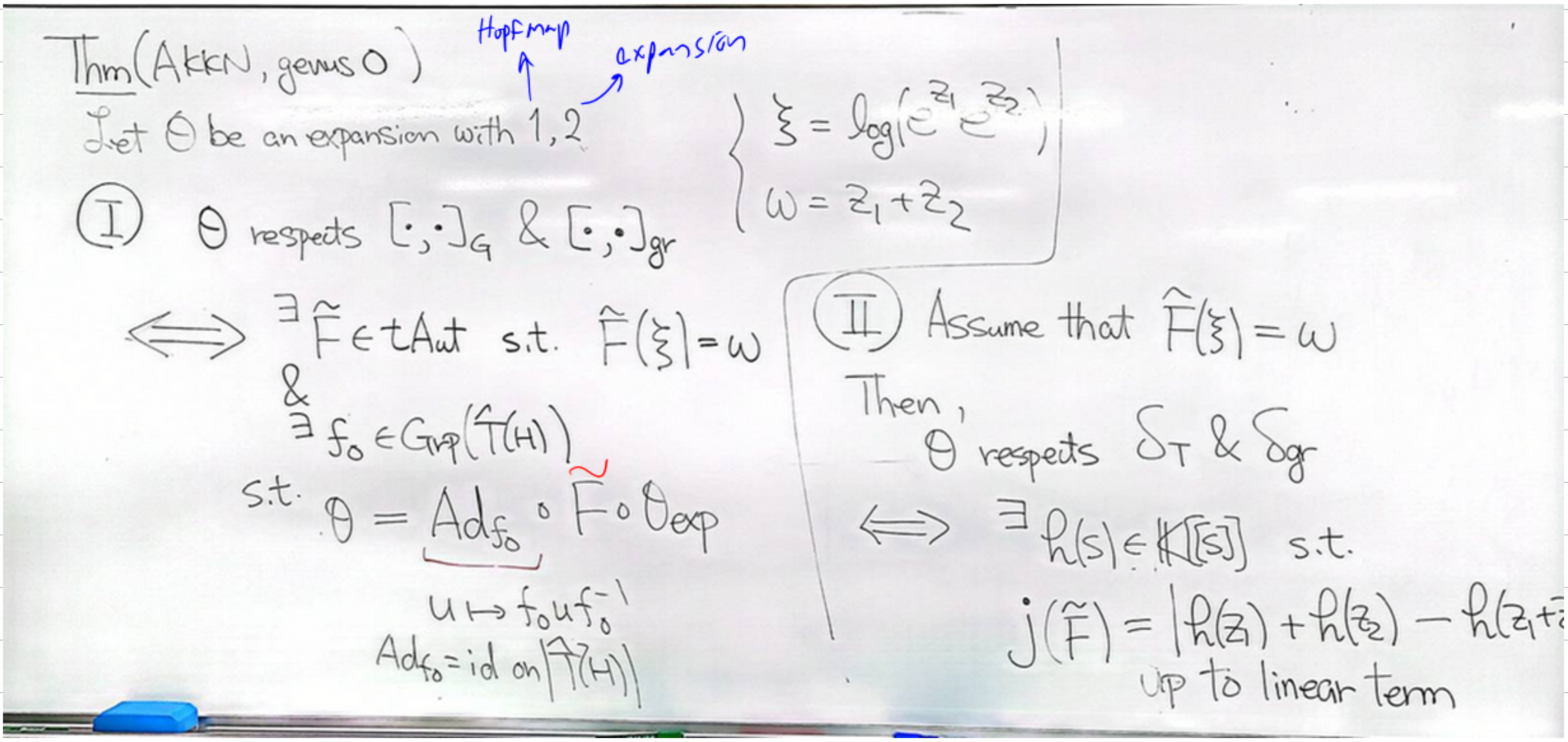
$\theta_{\exp} : \widehat{\mathbb{K}\pi} \longrightarrow \hat{T}(H)$

$\gamma_i \longmapsto \exp(z_i)$

Note:

$$tAut^+ := \left\{ \tilde{F} = (F, f_1, f_2) \;\middle|\; \begin{matrix} F \in Aut^+(\hat{T}(H), \Delta), \; f_1, f_2 \in \mathrm{Grp}(\hat{T}(H)) \\ F(z_j) = f_j^{-1} z_j f_j \end{matrix} \right\}$$

$\rightsquigarrow \forall \theta$ with 1, 2, 2.5 $\exists \tilde{F} \in tAut^+$ $\theta = F \circ \theta_{exp}$.



$\underline{\text{Thm}}$ (AKKN, genus 0)

Let Θ be an expansion with 1, 2 (Hopf map) (expansion)

$$\left\{ \begin{aligned} \xi &= \log(e^{z_1} e^{z_2}) \\ \omega &= z_1 + z_2 \end{aligned} \right.$$

Ⓘ Θ respects $[\cdot,\cdot]_G$ & $[\cdot,\cdot]_{gr}$

$\iff {}^{\exists}\tilde{F} \in tAut$ s.t. $\hat{F}(\xi) = \omega$

&

${}^{\exists} f_0 \in Grp(\hat{T}(H))$

s.t. $\Theta = \underbrace{Ad_{f_0}} \circ \tilde{F} \circ \theta_{exp}$

$u \mapsto f_0 u f_0^{-1}$

$Ad_{f_0} = id$ on $|\hat{T}(H)|$

Ⓘⓘ Assume that $\hat{F}(\xi) = \omega$

Then,

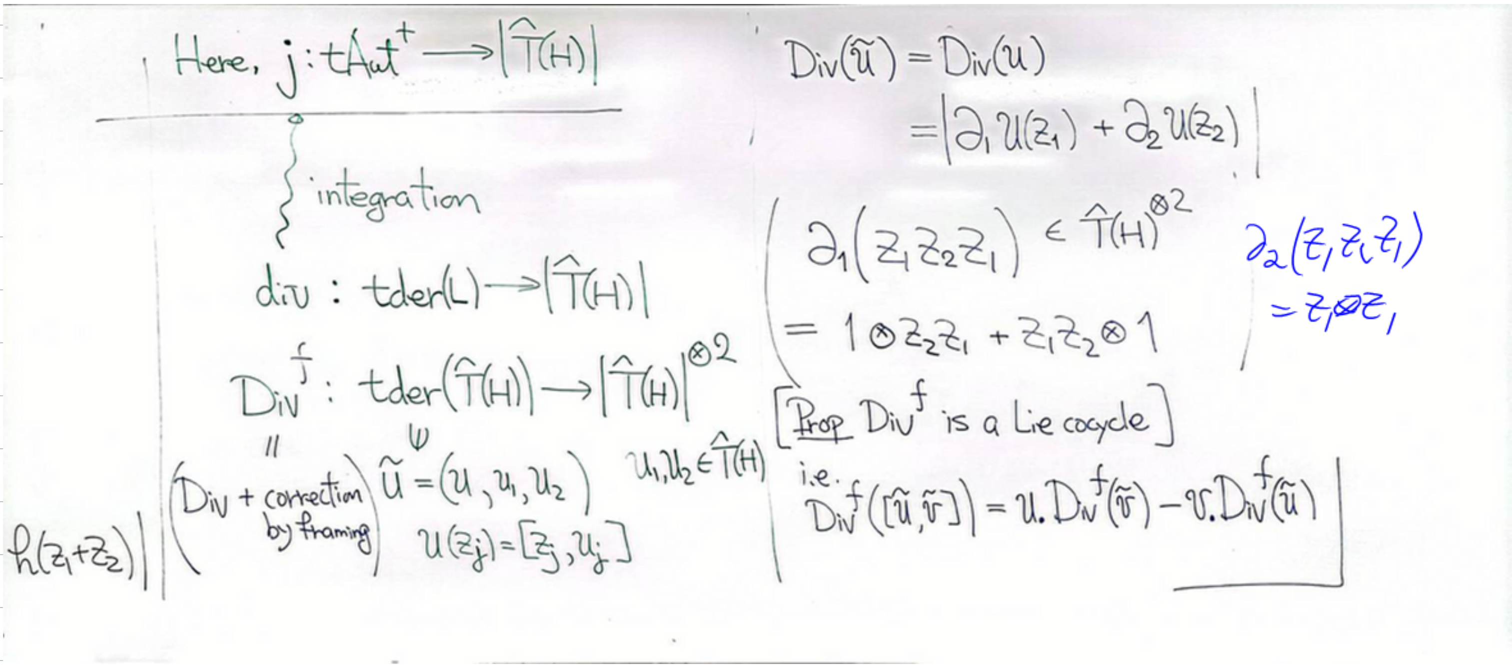
Θ respects δ_T & δ_{gr}

$\iff {}^{\exists} h(s) \in \mathbb{K}[[s]]$ s.t.

$j(\tilde{F}) = |h(z_1) + h(z_2) - h(z_1+z$

up to linear term

Q. why bother w/ Ad_{f_0}? Aren't inner automophisms anyway tangential?



Here, $j : tAut^+ \longrightarrow |\hat{T}(H)|$

$\Big\}$ integration

$div : tder(L) \longrightarrow |\hat{T}(H)|$

$Div^f : tder(\hat{T}(H)) \longrightarrow |\hat{T}(H)|^{\otimes 2}$

$\quad \overset{\Psi}{\tilde{u}} = (u, u_1, u_2) \quad u_1, u_2 \in \hat{T}(H)$

$\quad \| \quad u(z_j) = [z_j, u_j]$

$\left(Div + \text{correction by framing}\right)$

$h(z_1+z_2)|$

$Div(\tilde{u}) = Div(u)$

$= |\partial_1 u(z_1) + \partial_2 u(z_2)|$

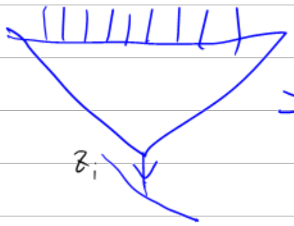
$\left(\begin{aligned} &\partial_1(z_1 z_2 z_1) \in \hat{T}(H)^{\otimes 2} \\ &= 1 \otimes z_2 z_1 + z_1 z_2 \otimes 1 \end{aligned} \right)$

$\partial_2(z_1 z_1 z_1) = z_1 \otimes z_1$

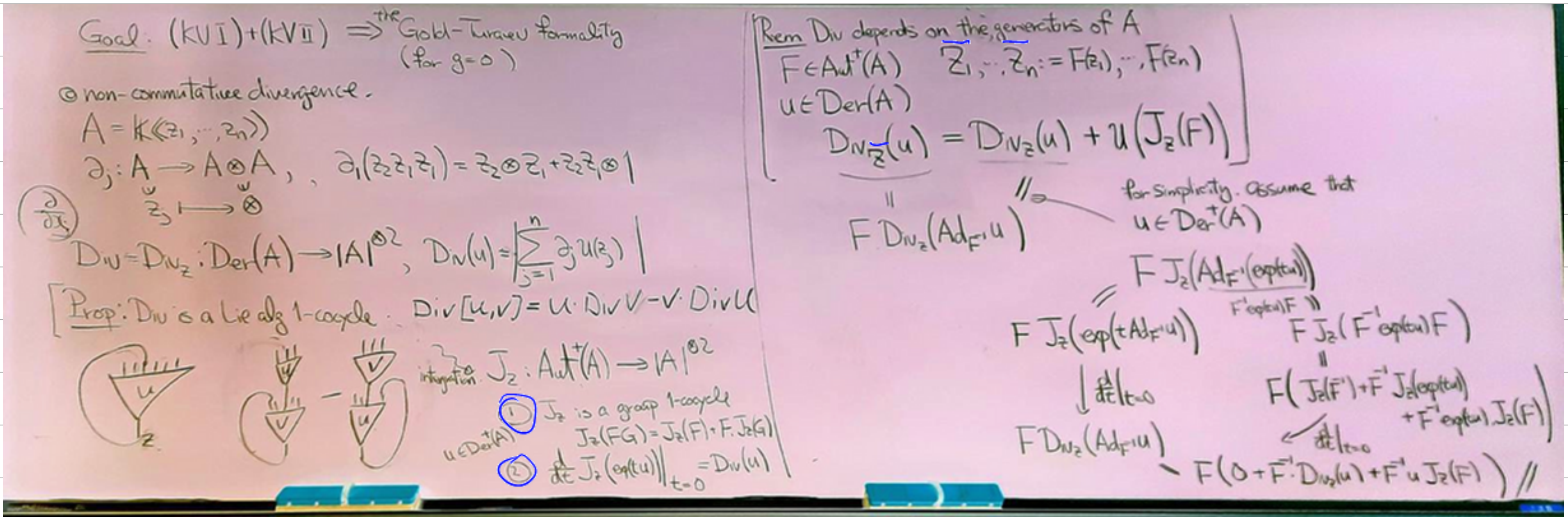
[$\underline{\text{Prop}}$ Div^f is a Lie cocycle]

i.e. $Div^f([\tilde{u}, \tilde{v}]) = u.Div^f(\tilde{v}) - v.Div^f(\tilde{u})$

$u(z_i) = [z_i, u_i]$



$=$



Goal: (KV I)+(KV II) $\Rightarrow$ the Gold-Turaev formality (for $g=0$)

⊙ non-commutative divergence.

$A = \mathbb{K}\langle\langle z_1, \cdots, z_n \rangle\rangle$

$\partial_j : A \longrightarrow A \otimes A$, $\partial_1(z_2 z_1 z_1) = z_2 \otimes z_1 + z_2 z_1 \otimes 1$

$\left(\frac{\partial}{\partial x_j}\right)$ $z_j \longmapsto \otimes$

$\mathrm{Div} = \mathrm{Div}_z : \mathrm{Der}(A) \longrightarrow |A|^{\otimes 2}$, $\mathrm{Div}(u) = \left|\sum_{j=1}^{n} \partial_j u(z_j)\right|$

[Prop: Div is a Lie alg 1-cocycle. $\mathrm{Div}[u,v] = u \cdot \mathrm{Div}\, v - v \cdot \mathrm{Div}\, u$

integration $J_z : \mathrm{Aut}^+(A) \longrightarrow |A|^{\otimes 2}$

① J_z is a group 1-cocycle $J_z(FG) = J_z(F) + F.J_z(G)$

$u \in \mathrm{Der}^+(A)$ ② $\frac{d}{dt} J_z(\exp(tu))\big|_{t=0} = \mathrm{Div}(u)$

[Rem Div depends on the generators of A

$F \in \mathrm{Aut}^+(A)$ $\bar{z}_1, \cdots, \bar{z}_n := F(z_1), \cdots, F(z_n)$

$u \in \mathrm{Der}(A)$

$\mathrm{Div}_{\bar{z}}(u) = \mathrm{Div}_z(u) + u(J_z(F))$]

$\mathrm{Div}_{\bar{z}}(u) = F\, \mathrm{Div}_z(\mathrm{Ad}_{F^{-1}} u)$

for simplicity, assume that $u \in \mathrm{Der}^+(A)$

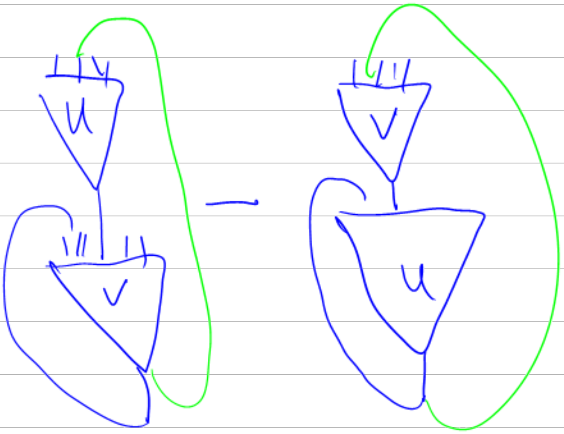
$F\, J_z(\mathrm{Ad}_{F^{-1}}(\exp(tu))) = F\, J_z(\exp(t\,\mathrm{Ad}_{F^{-1}} u))$

$\mathrm{Ad}_{F^{-1}}(\exp(tu)) = F^{-1}\exp(tu) F$

$F\, J_z(F^{-1}\exp(tu)F) = F\left(J_z(F^{-1}) + F^{-1} J_z(\exp(tu)) + F^{-1}\exp(tu)\, J_z(F)\right)$

$F\, J_z(\exp(t\,\mathrm{Ad}_{F^{-1}} u)) \xrightarrow{\frac{d}{dt}|_{t=0}} F\, \mathrm{Div}_z(\mathrm{Ad}_{F^{-1}} u)$

$\xrightarrow{\frac{d}{dt}|_{t=0}} F(0 + F^{-1}\mathrm{Div}_z(u) + F^{-1} u\, J_z(F))$ //



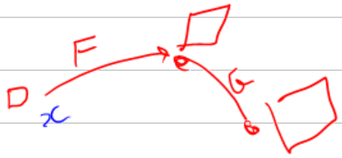
① $J_z(FG) = J_z(F) + F \cdot J_z(G)$

② $\frac{d}{dt} J(e^{tu})\big|_{t=0} = \mathrm{Div}(u)$

X is a v.f. on ~~M~~ $\mathbb{R}^n$ $X = \sum u_i \partial_{x_i}$

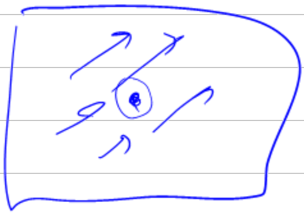
$\mathrm{div}\, X = \sum_i \partial_i u_i$ $F: \mathbb{R}^n \to \mathbb{R}^n$

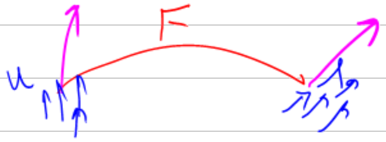
$\mathrm{Jac}(F): \mathbb{R}^n \to \mathbb{R}^n$

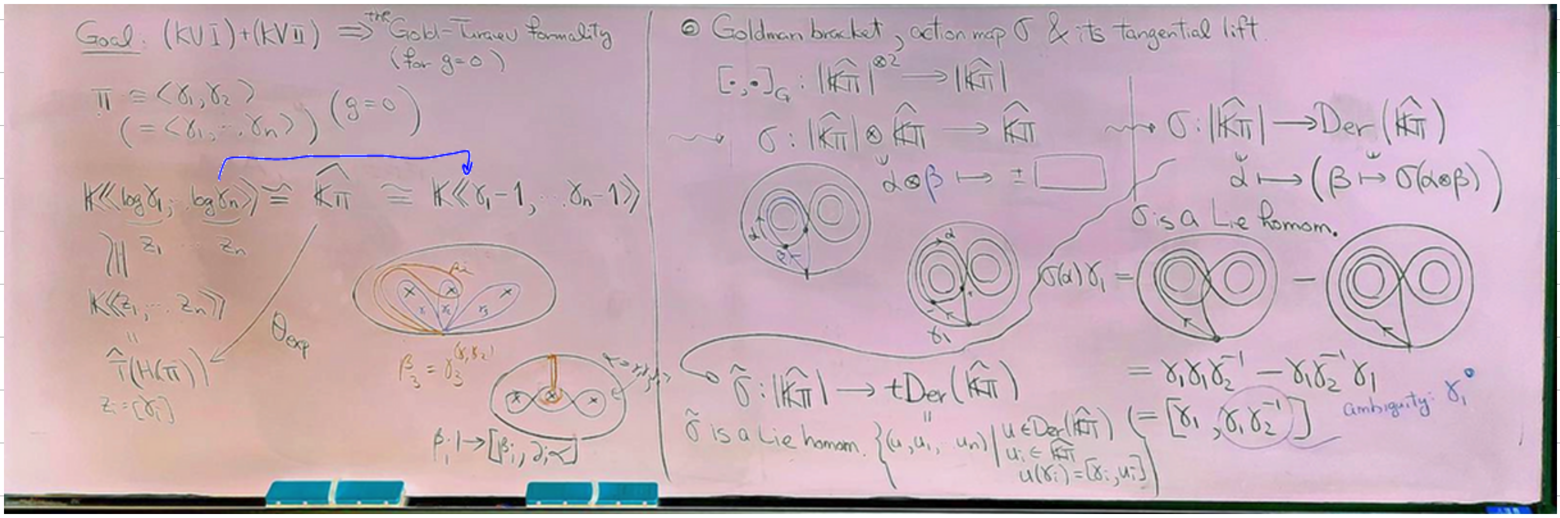


1. $\mathrm{Jac}(F /\!/ G) = \mathrm{Jac}(F) \circ (F\, \mathrm{Jac}(G))$

2. $\mathrm{Jac}(e^{tX})\big|_t = \mathrm{div}\, X$







Goal: (KV I)+(KV II) $\Rightarrow$ the Gold-Turaev formality (for $g=0$)

$\pi \cong \langle \gamma_1, \gamma_2 \rangle$ $(= \langle \gamma_1, \cdots, \gamma_n \rangle)$ $(g=0)$

$\mathbb{K}\langle\langle \log\gamma_1, \cdots, \log\gamma_n \rangle\rangle \cong \widehat{\mathbb{K}\pi} \cong \mathbb{K}\langle\langle \gamma_1 - 1, \cdots, \gamma_n - 1 \rangle\rangle$

$z_1 \cdots z_n$

$\mathbb{K}\langle\langle z_1, \cdots, z_n \rangle\rangle = \hat{T}(H(\pi))$ $\xleftarrow{\theta_{\exp}}$

$z_i = [\gamma_i]$

$\beta_3 = \gamma_3^{(\gamma_1\gamma_2)}$

$\beta_i \mapsto [\beta_i, \partial_i \sim]$

$\sim \gamma_1\gamma_2\gamma_3$

⊙ Goldman bracket, action map σ & its tangential lift.

$[\cdot,\cdot]_G : |\widehat{\mathbb{K}\pi}|^{\otimes 2} \longrightarrow |\widehat{\mathbb{K}\pi}|$

$\rightsquigarrow \sigma : |\widehat{\mathbb{K}\pi}| \otimes \widehat{\mathbb{K}\pi} \longrightarrow \widehat{\mathbb{K}\pi}$

$\alpha \otimes \beta \longmapsto \pm \boxed{}$

$\rightsquigarrow \sigma : |\widehat{\mathbb{K}\pi}| \longrightarrow \mathrm{Der}(\widehat{\mathbb{K}\pi})$

$\alpha \longmapsto (\beta \mapsto \sigma(\alpha \otimes \beta))$

σ is a Lie homom.

$\sigma(\alpha)\gamma_1 =$ ... $-$ ...

$= \gamma_1\gamma_1\gamma_2^{-1} - \gamma_1\gamma_2^{-1}\gamma_1$

$(= [\gamma_1, \gamma_1\gamma_2^{-1}]$

ambiguity: γ_1°

$\tilde{\sigma} : |\widehat{\mathbb{K}\pi}| \longrightarrow t\mathrm{Der}(\widehat{\mathbb{K}\pi})$

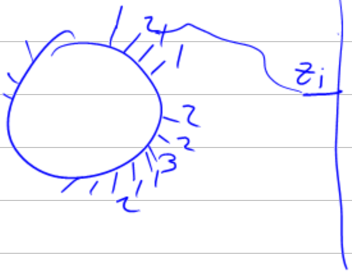
$\tilde{\sigma}$ is a Lie homom. $\left\{(u, u_1, \cdots, u_n) \,\middle|\, \begin{array}{l} u \in \mathrm{Der}(\widehat{\mathbb{K}\pi}) \\ u_i \in \widehat{\mathbb{K}\pi} \\ u(\gamma_i) = [\gamma_i, u_i] \end{array}\right\}$

$\log(1-x) = -\sum \frac{x^n}{n}$

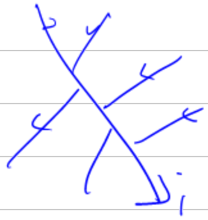
$z_i = \log(\gamma_i) = \log(1-(1-\gamma_i)) = -\sum (1-\gamma_i)^n/n$

$u \in |A|$ $\quad \sigma : |A| \to \mathrm{tder}$ $\quad \sigma(u) = (u_1, \dots, u_n)$

$\sigma(u)(z_i) = [z_i, u_i]$

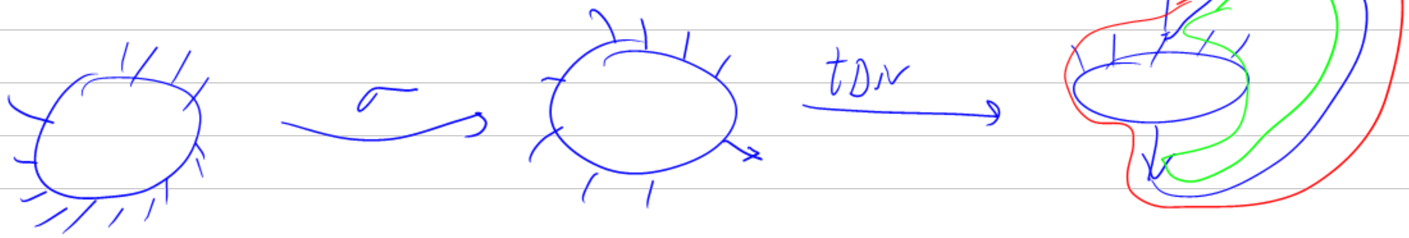


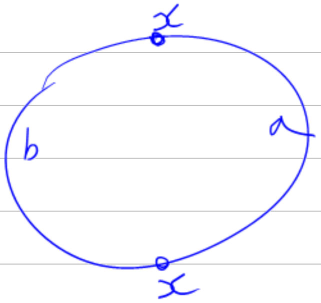
$\mathcal{A}^u \xrightarrow{\alpha} \mathcal{A}^w$ at tree level



$u_i := \partial_i u$

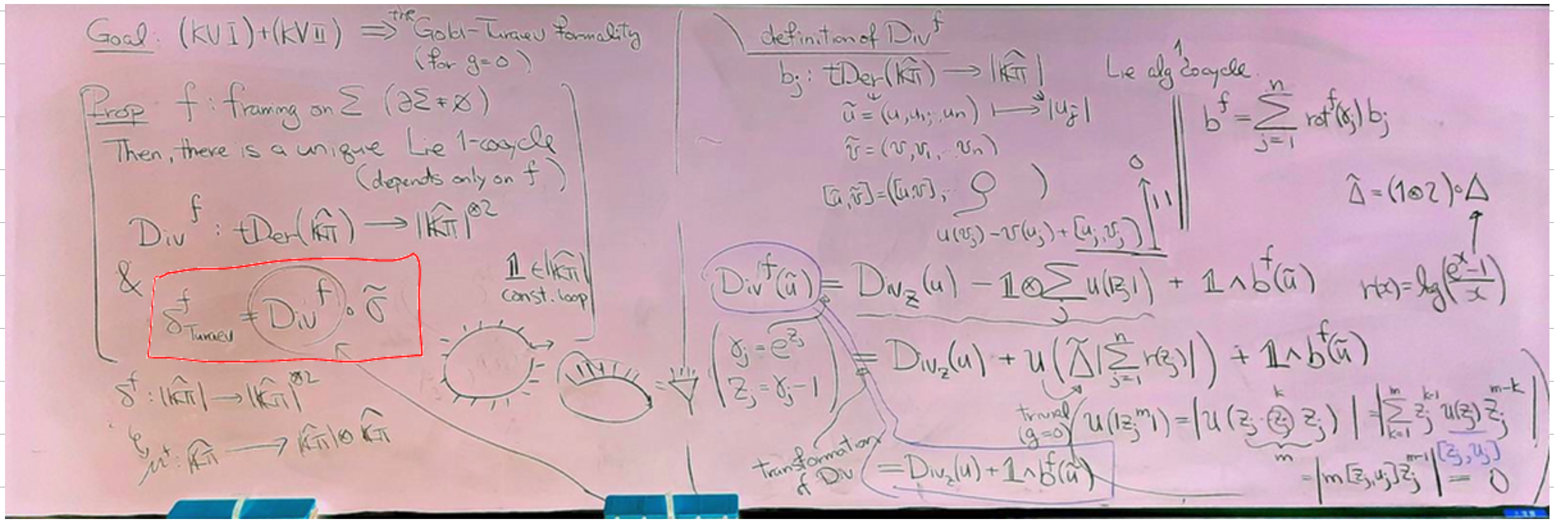
$\mathrm{div}(T) = \phi_u(T) - \phi_l(T)$





$\xrightarrow{\mathrm{Div}} a \otimes b + b \otimes a$

$\xrightarrow{\mathrm{dDiv}} ax \otimes b - a \otimes bx + bx \otimes a - b \otimes ax$



Goal: (KV I)+(KV II) $\Longrightarrow$ the Gold-Turaev formality (for g=0)

Prop f: framing on Σ $(\partial\Sigma \neq \emptyset)$
Then, there is a unique Lie 1-cocycle (depends only on f)

$$\mathrm{Div}^f : t\mathrm{Der}(\widehat{K\pi}) \longrightarrow |\widehat{K\pi}|^{\otimes 2}$$

& $\boxed{\delta^f_{\mathrm{Turaev}} = \mathrm{Div}^f \circ \tilde{\sigma}}$

$\mathbb{1} \in |\widehat{K\pi}|$ const. loop

$\delta^f : |\widehat{K\pi}| \longrightarrow |\widehat{K\pi}|^{\otimes 2}$

$\mu^f : \widehat{K\pi} \longrightarrow \widehat{K\pi} \hat{\otimes} \widehat{K\pi}$

definition of Div^f

$b_j : t\mathrm{Der}(\widehat{K\pi}) \longrightarrow |\widehat{K\pi}|$ Lie alg 1-cocycle

$\tilde{u} = (u, u_1, \dots, u_n) \longmapsto |u_j|$

$\tilde{v} = (v, v_1, \dots, v_n)$

$[\tilde{u}, \tilde{v}] = ([u,v], \dots \quad)$

$u(v_j) - v(u_j) + [u_j, v_j]$

$$b^f = \sum_{j=1}^{n} \mathrm{rot}^f(\gamma_j)\, b_j$$

$\hat{\Delta} = (1 \otimes \iota) \circ \Delta$

$r(x) = \log\left(\frac{e^x - 1}{x}\right)$

$$\mathrm{Div}^f(\tilde{u}) = \mathrm{Div}_z(u) - \mathbb{1} \otimes \sum_j u(|z_j|) + \mathbb{1} \wedge b^f(\tilde{u})$$

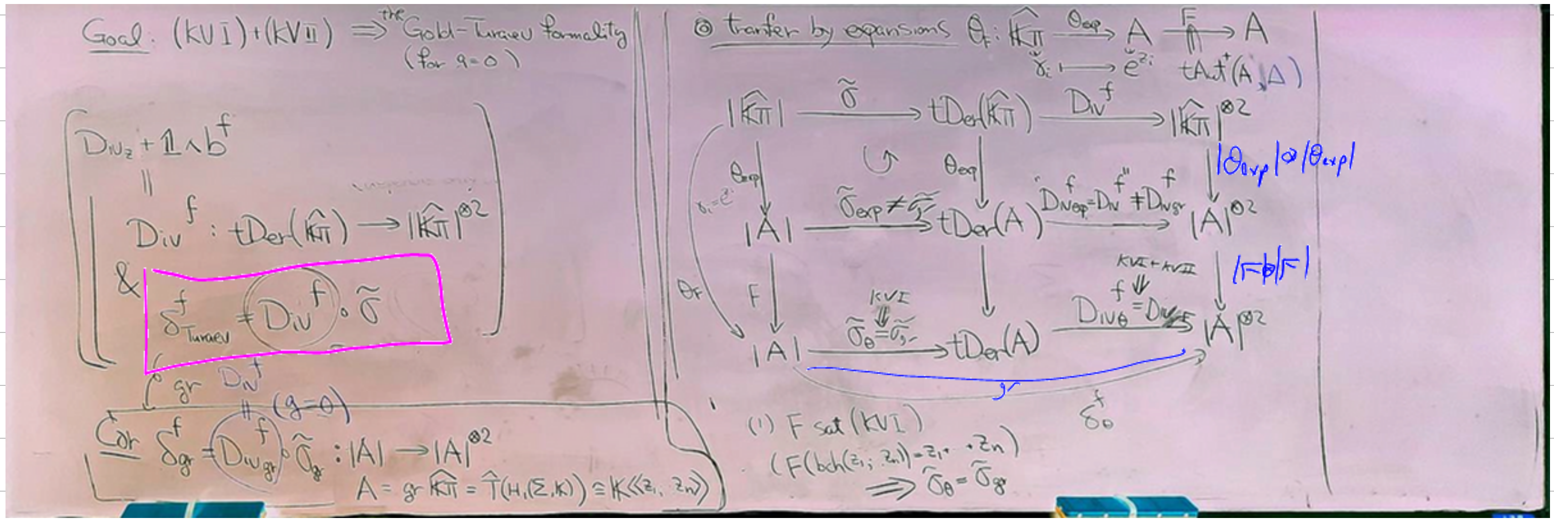
$\begin{pmatrix} \gamma_j = e^{z_j} \\ z_j = \gamma_j - 1 \end{pmatrix}$

$$= \mathrm{Div}_z(u) + u\Big(\tilde{\Delta}\big|\sum_{j=1}^{n} r(z_j)\big|\Big) + \mathbb{1} \wedge b^f(\tilde{u})$$

trivial (g=0) $u(|z_j^m|) = |u(z_j \cdots \overset{k}{z_j} \cdots z_j)| = \Big|\sum_{k=1}^{m} z_j^{k-1} u(z_j) z_j^{m-k}\Big|$

$= |m [z_j, u_j] z_j^{m-1}| = 0$ $[z_j, u_j]$

transformation of Div $\boxed{= \mathrm{Div}_z(u) + \mathbb{1} \wedge b^f(\tilde{u})}$



Goal: (KV I)+(KV II) $\Longrightarrow$ the Gold-Turaev formality (for g=0)

$\mathrm{Div}_z + \mathbb{1} \wedge b^f$
$\|$
$\mathrm{Div}^f : t\mathrm{Der}(\widehat{K\pi}) \longrightarrow |\widehat{K\pi}|^{\otimes 2}$

& $\boxed{\delta^f_{\mathrm{Turaev}} = \mathrm{Div}^f \circ \tilde{\sigma}}$

gr Div^f (g=0)

Cor $\delta^f_{gr} = \mathrm{Div}^f_{gr} \circ \tilde{\sigma}_{gr} : |A| \longrightarrow |A|^{\otimes 2}$

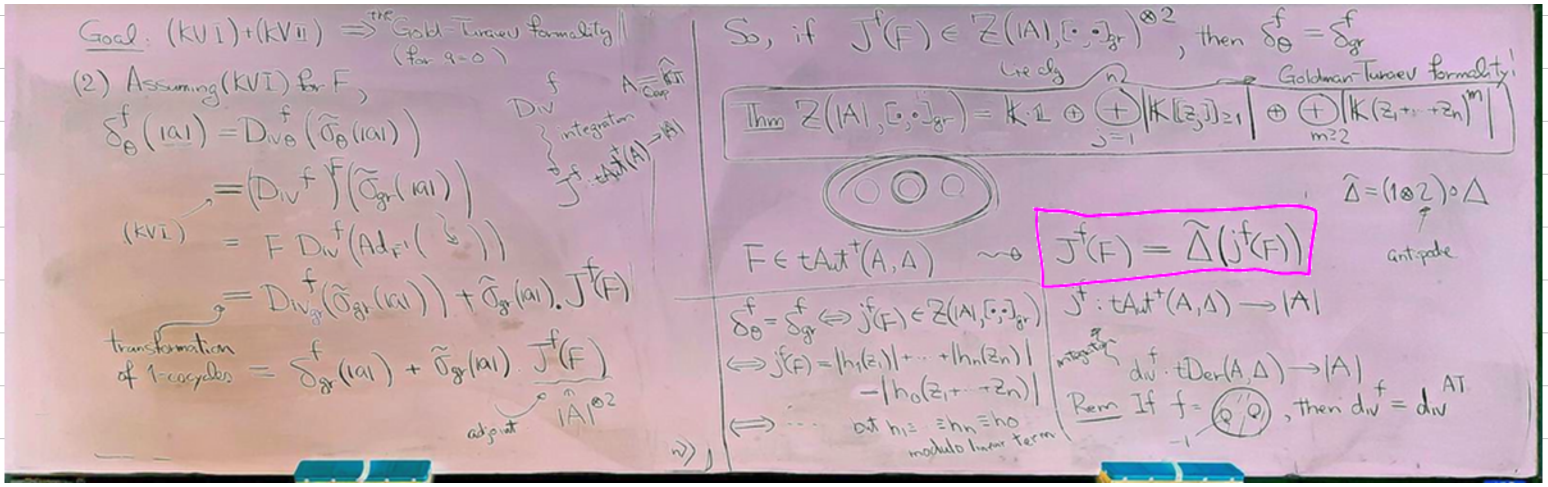
$A = \mathrm{gr}\,\widehat{K\pi} = \hat{T}(H_1(\Sigma, \mathbb{K})) \cong \mathbb{K}\langle\langle z_1, \dots, z_n \rangle\rangle$

⊚ tranfer by expansions $\theta_F : \widehat{K\pi} \xrightarrow{\theta_{\exp}} A \xrightarrow{F} A$

$\gamma_i \longmapsto e^{z_i}$ $\quad F \in tAut^+(A, \Delta)$

$$\begin{tikzcd}
{|\widehat{K\pi}|} \arrow[r, "\tilde{\sigma}"] \arrow[d, "\theta_{\exp}"', "\gamma_i = e^{z_i}"] \arrow[dd, bend right=60, "\theta_F"'] \arrow[rd, phantom, "\circlearrowright"] & t\mathrm{Der}(\widehat{K\pi}) \arrow[r, "\mathrm{Div}^f"] \arrow[d, "\theta_{\exp}"] & {|\widehat{K\pi}|^{\otimes 2}} \arrow[d, "|\theta_{\exp}| \otimes |\theta_{\exp}|"] \\
{|A|} \arrow[r, "\tilde{\sigma}_{\exp} \neq \tilde{\sigma}_{gr}"] \arrow[d, "F"] & t\mathrm{Der}(A) \arrow[r, "\mathrm{Div}^f_{\exp} = \mathrm{Div}^{f''} \neq \mathrm{Div}^f_{gr}"] \arrow[d] & {|A|^{\otimes 2}} \arrow[d, "|F| \otimes |F|"] \\
{|A|} \arrow[r, "\overset{KV I}{\tilde{\sigma}_\theta = \tilde{\sigma}_{gr}}"] \arrow[rr, bend right, "\delta^f_\theta"', "gr"] & t\mathrm{Der}(A) \arrow[r, "\overset{KV I + KV II}{\mathrm{Div}^f_\theta = \mathrm{Div}^f_{gr}}"] & {|A|^{\otimes 2}}
\end{tikzcd}$$

(1) F sat (KV I)
$(F(\mathrm{bch}(z_1, \dots, z_n)) = z_1 + \dots + z_n)$
$\Longrightarrow \tilde{\sigma}_\theta = \tilde{\sigma}_{gr}$



Goal: (KV I)+(KV II) $\Longrightarrow$ the Gold-Turaev formality (for g=0)

(2) Assuming (KV I) for F,

Div^f $\quad A \underset{\mathrm{Cap}}{=} \widehat{K\pi}$
integration
$J^f : tAut^+(A) \to |A|$

$$\delta^f_\theta(|a|) = \mathrm{Div}^f_\theta(\tilde{\sigma}_\theta(|a|))$$

$$= (\mathrm{Div}^f)^F(\tilde{\sigma}_{gr}(|a|))$$

(KV I)

$$= F\, \mathrm{Div}^f(\mathrm{Ad}_{F^{-1}}(\quad))$$

$$= \mathrm{Div}^f_{gr}(\tilde{\sigma}_{gr}(|a|)) + \tilde{\sigma}_{gr}(|a|).J^f(F)$$

transformation of 1-cocycles $= \delta^f_{gr}(|a|) + \tilde{\sigma}_{gr}(|a|).\underbrace{J^f(F)}$

adjoint $\quad |\hat{A}|^{\otimes 2}$

So, if $J^f(F) \in Z(|A|, [\cdot,\cdot]_{gr})^{\otimes 2}$, then $\delta^f_\theta = \delta^f_{gr}$ Goldman-Turaev formality!

Lie alg

$$\boxed{\text{Thm } Z(|A|, [\cdot,\cdot]_{gr}) = \mathbb{K}\cdot\mathbb{1} \oplus \bigoplus_{j=1}^{n} |\mathbb{K}[[z_j]]_{\geq 1}| \oplus \bigoplus_{m \geq 2} |\mathbb{K}(z_1 + \dots + z_n)^m|}$$

$\hat{\Delta} = (1 \otimes \iota) \circ \Delta$
antipode

$F \in tAut^+(A, \Delta) \rightsquigarrow \boxed{J^f(F) = \hat{\Delta}(j^f(F))}$

$j^f : tAut^+(A, \Delta) \longrightarrow |A|$

integration

$\mathrm{div}^f : t\mathrm{Der}(A, \Delta) \longrightarrow |A|$

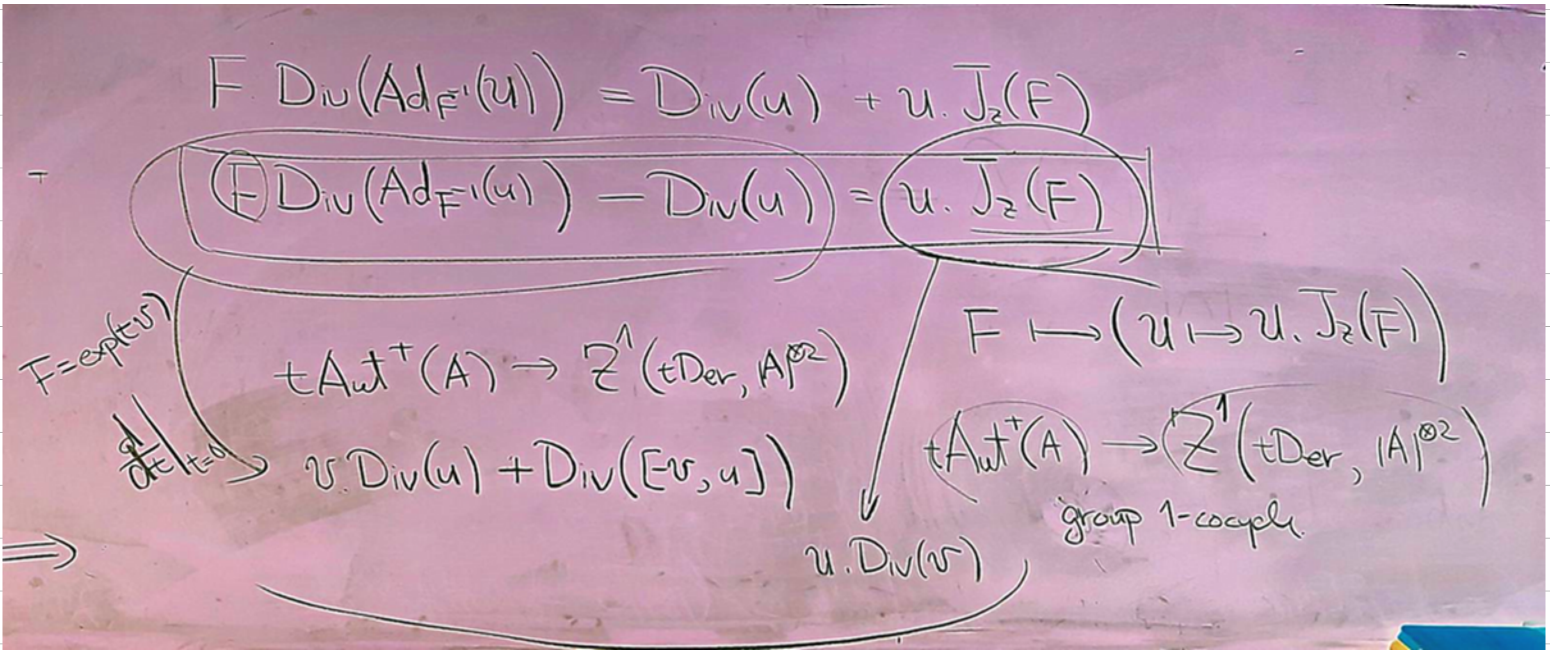
Rem If f = , then $\mathrm{div}^f = \mathrm{div}^{AT}$

-1

$\delta^f_\theta = \delta^f_{gr} \Longleftrightarrow j^f(F) \in Z(|A|, [\cdot,\cdot]_{gr})$

$\Longleftrightarrow j^f(F) = |h_1(z_1)| + \dots + |h_n(z_n)| - |h_0(z_1 + \dots + z_n)|$

$\Longleftrightarrow \cdots$ but $h_1 \equiv \dots \equiv h_n \equiv h_0$ modulo linear term



$$F \cdot \mathrm{Div}(\mathrm{Ad}_{F^{-1}}(u)) = \mathrm{Div}(u) + u \cdot \overline{J}_2(F)$$

$$\boxed{F\,\mathrm{Div}(\mathrm{Ad}_{F^{-1}}(u)) - \mathrm{Div}(u) = u \cdot \overline{J}_2(F)}$$

$F = \exp(tv)$

$$tAut^+(A) \to Z^1(tDer, |A|^{\otimes 2})$$

$$\left.\frac{d}{dt}\right|_{t=0} \to v \cdot \mathrm{Div}(u) + \mathrm{Div}([-v, u])$$

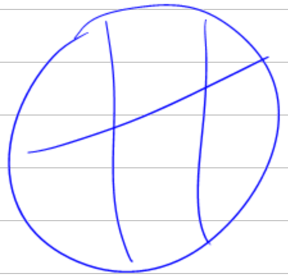
$\Longrightarrow$

$$u \cdot \mathrm{Div}(v)$$

$$F \longmapsto (u \mapsto u \cdot J_2(F))$$

$$tAut^+(A) \to Z^1(tDer, |A|^{\otimes 2})$$

group 1-cocycle



$$\underset{\delta}{\overset{\|}{\sigma}} /\!/ \mathrm{div} \; : |A| \longrightarrow |A| \otimes |A|$$

$$[a\,b\,\overbrace{b\,c\,c\,a\,a\,b}\,b] \longrightarrow \sum [c\,c\,a\,\overset{+b}{a}] \otimes [b\,a\,\overset{+b}{b}]$$

$$\underbrace{\underbrace{e^{-\partial_2} b\, e^{\partial_2}}\, e^{-\partial_3} c\, e^{\partial_3}}$$

$$\partial_i \partial_j : |A| \longrightarrow A \otimes A$$

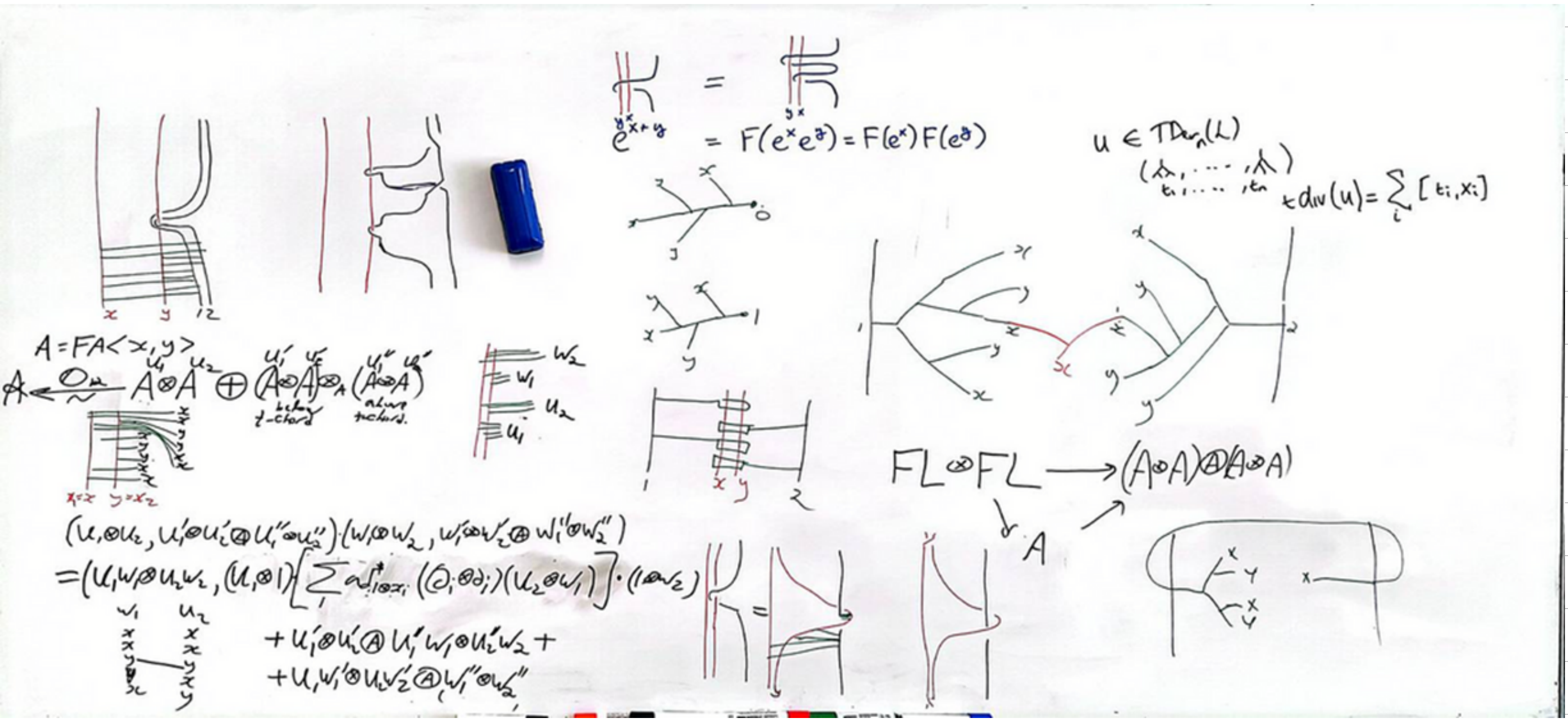
$$[\,\_\,|\,\_\,|\,\_\,]$$

$$\underbrace{\left| \sum \alpha_{ij} \partial_i \partial_j \right|}$$

$$\alpha_{ij} \in A^{\otimes 2}$$

$$\alpha_{ij} = \delta_{ij} + \dots$$

$$\left| \sum \alpha_{ij} \partial_i \partial_j \right| \overset{?}{=} \left| \sum \beta_{ij} \partial_i \partial_j \right|$$



$$e^{x+y} = F(e^xe^y)=F(e^x)F(e^y)$$

$u \in TDer_n(L)$

$(\lambda_1, \dots, \lambda_n)$
$t_1, \dots, t_n$ $t\mathrm{div}(u)=\sum_i [t_i,x_i]$

$A=FA\langle x,y\rangle$

$$A \xleftarrow[\sim]{\oplus\mu} \overset{u_1}{A}\otimes\overset{u_2}{A} \oplus (\overset{u_1'}{A}\otimes\overset{u_2'}{A})\otimes_A(\overset{u_1''}{A}\otimes\overset{u_2''}{A})$$

$FL\otimes FL \longrightarrow (A\otimes A)\otimes_A(A\otimes A)$

$\searrow A \nearrow$

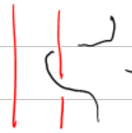
$$(u_1\otimes u_2, u_1'\otimes u_2'\otimes_A u_1''\otimes u_2'')\cdot(w_1\otimes w_2, w_1'\otimes w_2'\otimes_A w_1''\otimes w_2'')$$

$$=(u_1w_1\otimes u_2w_2, (u_1\otimes 1)\Big[\sum_i ad^{*}_{1\otimes x_i}((\partial_i\otimes\partial_i)(u_2\otimes w_1))\Big]\cdot(1\otimes w_2)$$

$$+u_1'\otimes u_2'\otimes_A u_1'' w_1\otimes u_2'' w_2 +$$

$$+u_1w_1'\otimes u_2 w_2'\otimes_A w_1''\otimes w_2''$$

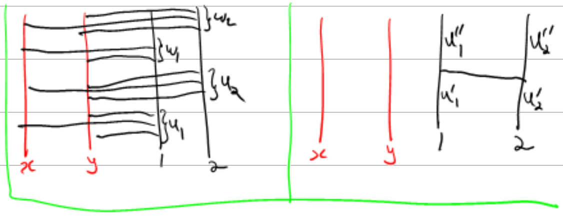
$F=(f_1,f_2)\longmapsto(f_1^{+\psi},f_2+\varphi)$ deg 7



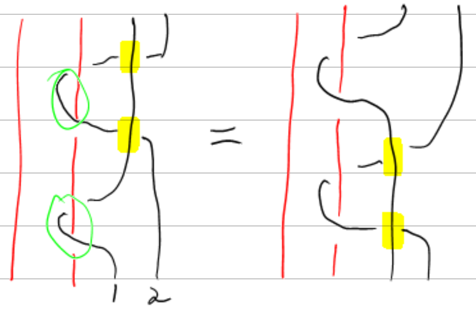
$$\longrightarrow F(e^y) \rightsquigarrow F(e^y)+\underbrace{[y,\varphi]}_{8}$$

(ψ,φ) satisfies KV1:

$[\psi,x]+[\varphi,y]=0$



How does a change as above effect the equation

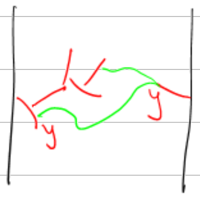


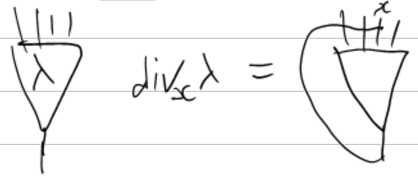
In s-degree 0: no effect

In s-degree 1, t-degree 9:

$$\begin{matrix}1\otimes 1\\ 1\otimes[y,\varphi]\end{matrix} - \begin{matrix}1\otimes[y,\varphi]\\ 1\otimes 1\end{matrix} - \begin{matrix}1\otimes 1\\ [y,\varphi]\otimes 1\end{matrix} + \begin{matrix}[y,\varphi]\otimes 1\\ 1\otimes 1\end{matrix}$$

$$[[y,\varphi]\otimes 1, 1\otimes y]-[1\otimes[y,\varphi], y\otimes 1] \underset{s\text{-deg }0}{=} 0 \quad +$$





$$\mathrm{div}(\psi,\varphi)=\mathrm{div}_x\psi+\mathrm{div}_y\varphi$$

$$\mathrm{div}_x\lambda+\mathrm{div}_y\lambda=0$$